Preference for a Vanishingly Small Cosmological Constant in Supersymmetric Vacua in a Type IIB String Theory Model

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(Dated: November 30, 2012)

We study the probability distribution $P(\Lambda)$ of the cosmological constant Λ in a specific set of KKLT type models of supersymmetric IIB vacua. $P(\Lambda)$ is divergent at $\Lambda = 0^-$ and the likely value of Λ drops exponentially as the number of complex structure moduli $h^{2,1}$ increases. Also, owing to the hierarchical and approximate no-scale structure, the probability of having a positive Hessian (mass squared matrix) approaches unity as $h^{2,1}$ increases.

INTRODUCTION

Since string theory has many solutions, we are interested in the probability distribution $P(\Lambda)$ of the cosmological constant Λ of the solutions. In this paper, we consider the supersymmetric vacua of the KKLT type in Type IIB string theory [1, 2]. To be specific, the model of interest in this paper is given by (setting Planck scale $M_P = 1$)

$$\begin{aligned} V =& e^K \left(|D_T W|^2 + |D_S W|^2 + |D_{U_i} W|^2 - 3|W|^2\right), \\ K =& 3 \ln\left(T + \bar{T}\right) - \ln\left(S + \bar{S}\right) - \sum_{i=1}^{h^{2,1}} \ln\left(U_i + \bar{U}_i\right), \\ W =& W_0 + A e^{-aT}, \\ W_0 =& c_1 + \sum_{i=1}^{h^{2,1}} b_i U_i - \left(c_2 + \sum_{i=1}^{h^{2,1}} d_i U_i\right) S, \end{aligned} \tag{1}$$

where T is the Kähler modulus, S is the dilation and U_i are the complex structure moduli, while flux parameters b_i, c_j and d_i vary as the quantized fluxes vary [3–5]. Validity of the weak coupling approximation requires that $s = \mathrm{Re}\, S > 1$ while reality of the Kähler potential K requires that $u_i = \mathrm{Re}\, U_i > 0$. The second term (with parameters $a > 0$ and A) in the superpotential W is non-perturbative. The W_0 used here for the complex structure sector is motivated by the toroidal orbifolds of T^6 [6, 7].

After finding the supersymmetric vacuum solution, we can express Λ in terms of the parameters. Treating the parameters as random variables with some uniform (or smooth) distributions, we find $P(\Lambda)$.

- $P(\Lambda)$ diverges at $\Lambda = 0^-$ for the single Kähler modulus case : $P(\Lambda) \simeq (\ln|\Lambda|)^2/2e^{3/2}\sqrt{|\Lambda|}$, where only the parameter c_1 is treated as a random variable.

- $P(\Lambda)$ peaks at $\Lambda = 0^-$ for the model (1) with multi-complex structure moduli. In one scenario, the expected value $\langle|\Lambda|\rangle$ of Λ is shown in FIG. 1 as a function of $h^{2,1}$ (note that $h^{2,1} \sim \mathcal{O}(100)$ in many known models). Because of the long tail in $P(\Lambda)$, we see that $\langle|\Lambda|\rangle$ does not drop as $h^{2,1}$ increases. To get a better feeling of the property of $P(\Lambda)$, let us introduce $|\Lambda|_{Y\%}$, defined by $\int_{-|\Lambda|_{Y\%}}^{0} P(\Lambda)\, d\Lambda = Y\%$. That is, there is a $Y\%$ probability that Λ will fall in the range $0 \le |\Lambda| \le |\Lambda|_{Y\%}$. FIG. 1 shows the result of a specific uniform distribution of the flux parameters. Here, typical $|\Lambda|_{Y\%}$ decreases exponentially as $h^{2,1}$ increases. For the sake of discussion, if we take $|\Lambda|_{50\%}$ to be the likely value of $|\Lambda|$ and assume its behavior in FIG. 1 extends to larger $h^{2,1}$, we see that $|\Lambda| \simeq 10^{-0.82\, h^{2,1} + 2.7}$ for $h^{2,1} > 5$.

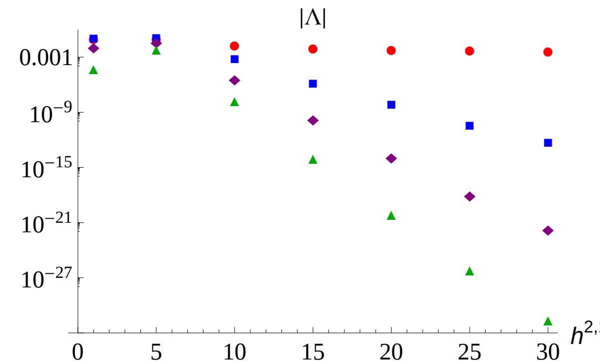

FIG. 1: The comparison of $\langle|\Lambda|\rangle$ (red circle), $|\Lambda|_{80\%}$ (blue square), $|\Lambda|_{50\%}$ (purple diamond), $|\Lambda|_{10\%}$ (green triangle) are shown as functions of the number of complex structure moduli $h^{2,1} = 1, 5, 10, 15, 20, 25, 30$.

- Another interesting property is the positivity of the Hessian $(\partial_i \partial_j V)$. Here we see that the probability of having a positive Hessian increases quickly towards unity as $h^{2,1}$ increases. This property is due to the approximate no-scale behavior and the hierarchical structure adopted. The latter is fully justified for large $h^{2,1}$.

This preference for a vanishingly small Λ is interesting since energetics alone may suggest more negative Λs should be preferred. This property is a consequence of the probability theory for the specific functional form of Λ in terms of the flux parameters [8]. This result

strengthens our earlier observation for positive Λ, where the functional form of Λ is only known approximately [8, 9]. That the Hessian is mostly positive allows us to be optimistic in the search for de-Sitter vacua in KKLT type models. We shall comment on what needs to be done to find a peaking $P(\Lambda)$ when supersymmetry is broken in KKLT models.

THE SINGLE KÄHLER MODULUS CASE

We first focus on the simple model for a single Kähler modulus stabilization at supersymmetric vacua as in [1],

$$K_K = -3\ln\left(T+\bar{T}\right), \quad W = W_0 + Ae^{-aT}. \tag{2}$$

where $W_0 = c_1$ here. Now we solve the supersymmetric condition $D_T W = \partial_T W + (\partial_T K)W = 0$. The imaginary part of T has a cosine type of potential and therefore the solution stays at $\mathrm{Im}\, T = 0$. Then the condition is written by

$$-3y = (2x+3)e^{-x} \tag{3}$$

where we defined $x = at = a\,\mathrm{Re}\, T$ and the parameter $y = W_0/A$. The real part of the modulus is defined to be positive so that the volume of compactification is correctly measured. Therefore the RHS of (3) is not only positive, but has an upper bound value of 3. The corresponding condition for the LHS implies $-1 < y \le 0$. If we rewrite in terms of $X = -x - 3/2$, (3) becomes

$$z \equiv \frac{3}{2}e^{-3/2}y = Xe^X. \tag{4}$$

where $z \simeq 0.3y$. The solution of this equation is known as a *Lambert* $\mathcal{W}$ function or a product logarithm. There are two branches of solutions : $\mathcal{W}_{-1}$ for $X \le -1$ and $\mathcal{W}_0$ for $X \ge -1$. Since t measures the volume of the 6-manifold for compactification, we need $X < -3/2$. This branch of solution can be expanded around $z \lesssim 0$ by

$$X(y) = \mathcal{W}_{-1}(z) = L_1(z) - L_2(z) + \cdots \tag{5}$$

where $L_1(z) = \ln(-z)$ and $L_2(z) = \ln[-\ln(-z)]$. So $x = -(X+3/2)$ is solved in terms of the parameter y. Inserting this into the potential $V = e^{K_K}(K_K^{T\bar{T}}|D_T W|^2 - 3|W|^2)$, we get

$$\begin{aligned}\Lambda \equiv V|_{\min} &= -3e^{K_K}|W|^2 \\ &= a^3A^2e^{3+2\mathcal{W}_{-1}(z)}/\left[9+6\mathcal{W}_{-1}(z)\right] \\ &\overset{y\to 0}{\simeq}\ a^3A^2\frac{3y^2}{8\ln(-y)}.\end{aligned} \tag{6}$$

We see that small Λ is precisely related to small y or W_0.

The stability of the vacuum can be understood easily. Using the condition (3) to eliminate A, we obtain

$$\partial_t^2 V|_{\min} = 3a^5W_0^2\frac{2x^2+5x+2}{2x^3(2x+3)^2}. \tag{7}$$

Since $x > 0$ and $a > 0$, the stability condition is automatically satisfied for this model (2).

The Probability Distribution of Λ - The probability distribution $P(\Lambda)$ is easy to obtain :

$$P(\Lambda) = \int_{-1}^{0} dy\, P(y)\,\delta\left(\frac{a^3A^2e^{3+2X}}{9+6X} - \Lambda\right) \tag{8}$$

where $P(y)$ is the distribution of $-1 < y \le 0$ and $X(y) = \mathcal{W}_{-1}(3e^{-3/2}y/2)$. If we set $A = 1$ for simplicity, y obeys the same distribution as that of W_0. The resulting probability distribution $P(\Lambda)$ is easy to evaluate numerically. It is shown in FIG 2, where $a = A = 1$ and W_0 has an uniform distribution in the range $-1 \le W_0 \le 0$. We see that it peaks at $\Lambda = 0^-$.

It is interesting to see analytically the peaking behavior of $P(\Lambda)$. Given $P(y)$, the integration in (8) can be easily performed using the property of delta function $\delta(g(y)) = |g'(y_0)|^{-1}\delta(y-y_0)$. So we need to express y as a function of Λ. That is, let us solve (6) for y. With $a = A = 1$, let us choose a uniform distribution for $-1 < y \le 1$. Introducing $p = -(3+2X)$, (6) becomes

$$-1/3\Lambda = p\,e^p. \tag{9}$$

Now the physical constraint for $X(y) = \mathcal{W}_{-1}(z)$ as in (5) requires p to satisfy $X = -(p+3)/2 < -3/2$. So the solution for $p > 0$ is given in terms of the other branch of the Lambert $\mathcal{W}$ function, namely,

$$\begin{aligned}p =& \mathcal{W}_0\left(-\frac{1}{3\Lambda}\right) = \sum_{m=1}^{\infty}\frac{(-m)^{m-1}}{m!}\left(-\frac{1}{3\Lambda}\right)^m \\ &\overset{\Lambda\sim 0}{\to} -\ln\left[3\Lambda\ln(-3\Lambda)\right] + \cdots\end{aligned} \tag{10}$$

Rewriting (6) using the basic relation of the Lambert $\mathcal{W}$ function, we have

$$\Lambda = \left(\frac{3}{2}e^{-3/2}y\right)^2\frac{e^3}{X^2(9+6X)}, \tag{11}$$

Since now $X(y)$ is related to $p = \mathcal{W}_0(-1/3\Lambda)$, we obtain, after using (9),

$$y = -\frac{1}{3}e^{-\mathcal{W}_0(-1/3\Lambda)/2}\left[\mathcal{W}_0\left(-\frac{1}{3\Lambda}\right)+3\right]. \tag{12}$$

where the overall sign of y is dictated by (3).

Using the solution (12), we get, after some calculations,

$$\begin{aligned}P(\Lambda) =& \frac{1}{2}\left[3+2\mathcal{W}_{-1}(f(\Lambda))\right]^2 e^{-3/2-\mathcal{W}_{-1}(f(\Lambda))}, \\ f(\Lambda) =& -\frac{1}{2}\left[3+\mathcal{W}_0\left(-\frac{1}{3\Lambda}\right)\right]e^{-3/2-\mathcal{W}_0(-1/3\Lambda)/2}.\end{aligned} \tag{13}$$

Using the expansion formulae of the $\mathcal{W}$ functions, we find

$$P(\Lambda) \overset{\Lambda\to 0^-}{\simeq} \frac{(3+\ln|\Lambda|)^2}{2e^{3/2}\sqrt{|\Lambda|}} + \cdots. \tag{14}$$

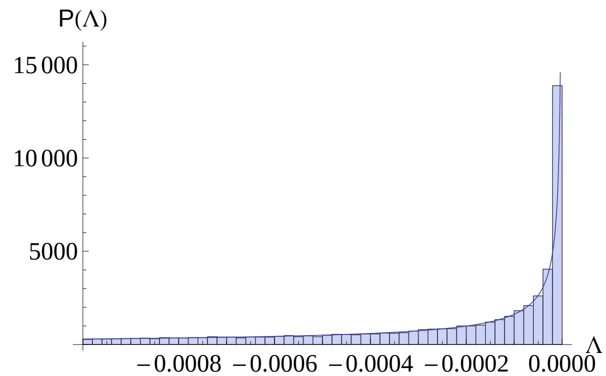

FIG. 2: The probability distribution $P(\Lambda)$ of the single Kähler modulus model (2). The (blue) bars are numerical result for (8) while the curve is for the analytical formula (13). Both agree nicely even at $\Lambda = 0^-$, where $P(\Lambda)$ is divergent.

Thus we see that $P(\Lambda)$ is actually divergent as $\Lambda \to 0^-$. We present both the "analytical result" (13) and the numerical result (8) in FIG 2. Even though $P(\Lambda)$ is divergent at $\Lambda = 0^-$, it is properly normalized in this paper, i.e., $\int P(\Lambda) d\Lambda = 1$.

SOLVING THE MODEL

Solutions - Now we are ready to solve the model (1) with $h^{2,1}$ number of complex structure moduli for supersymmetric vacua, i.e., $D_T W = D_S W = D_{U_i} W = 0$. Picking the Im $T =$ Im $S =$ Im $U_i = 0$ solution, we have

$$u_i = \frac{-1}{h^{2,1} - 2} \frac{\hat{c}_1 - s c_2}{b_i - s d_i}, \tag{15a}$$

$$(h^{2,1} - 2) \frac{\hat{c}_1 + s c_2}{\hat{c}_1 - s c_2} = \sum_{i=1}^{h^{2,1}} \frac{b_i + s d_i}{b_i - s d_i}, \tag{15b}$$

$$3W + 2xAe^{-x} = 0, \tag{15c}$$

$$W|_{\min} = -\frac{2(\hat{c}_1 + s c_2) \prod_{i=1}^{h^{2,1}} (b_i - s d_i)}{\sum_{i=1}^{h^{2,1}} (b_i + s d_i) \prod_{j \neq i} (b_i - s d_i)} \tag{15d}$$

where $s = \text{Re } S, u_i = \text{Re } U_i$, $x = a \, \text{Re } T = at$ and $\hat{c}_1 = c_1 + Ae^{-x}$. Note that (15b,c) form a pair of equations for s and x. Solving them determines s and x as well as u_i, so $W|_{\min}$ becomes a function of the parameters of the model.

To solve this efficiently, we assume that the complex structure and dilation sector are stabilized at higher energy scales than that of the Kähler modulus t. In this hierarchical setup, we first stabilize u_i and s in the model (1) and then the Kähler modulus stabilization can be dealt with. Under $\hat{c}_1 = c_1 + Ae^{-x} \to c_1$, $W|_{\min} \to W_0|_{\min}$, and (15a,b,d) reduce to those studied in [9] based on [7]. Although we deal with approximate solutions here, we can solve for all u_i even in the presence of the non-perturbative term Ae^{-x}. After determining $W_0|_{\min}$ and s, we solve for x and $W|_{\min}$. The validity of this hierarchical setup will be checked a posteriori.

Probability Distribution of Λ - Combining the previous analysis, the Λ of the supersymmetric vacuum is now determined to be

$$\Lambda = -3e^K |W|^2 = \frac{1}{2^{h^{2,1}+1} s \prod u_i} \frac{a^3 A^2 e^{3 + \mathcal{W}_{-1}(z)}}{9 + 6\mathcal{W}_{-1}(z)}. \tag{16}$$

It is now straightforward to calculate probability distribution $P(\Lambda)$ numerically, given the distributions of the parameters of the model (1). We see that $P(\Lambda)$ typically peaks sharply at $\Lambda = 0^-$.

To be specific, consider the following scenario. To deal with a divergent value of Λ coming from $1/u_i$, we introduce a cutoff f for flux inputs, $b_i = -f, -f \leq d_i \leq f$ while keeping $-1 \leq c_j \leq 1$ at each $h^{2,1}$ such that 90% of the Λ values fall within the Planck scale ($M_P = 1$) range, similarly to [9]. Note that the qualitative result will not change even if b_i is randomized, though this will take more computer time. The remaining 10% of the data sitting over the Plank scale are discarded to maintain the validity of this supergravity approximation. As before, we set $a = A = 1$. The result is shown in FIG. 1. We see that the likely $|\Lambda|$ drops exponentially as $h^{2,1}$ increases, even though $\langle |\Lambda| \rangle$ stays more or less constant. Similar qualitative properties are expected for some choices of distributions for the parameters, but not for others.

What happens when we turn on a supersymmetry breaking term? If this term is simply an additive term to the potential V in (1) and results in an additive term to Λ, then the peaking behavior of $P(\Lambda)$ will generically be erased [8]. To maintain and even enhance the peaking behavior in $P(\Lambda)$, the supersymmetry breaking term must couple to the rest of V. So a good understanding of the supersymmetry breaking dynamics of the KKLT scenario (including back-reactions) is crucial.

Probability of positive mass matrix - Since we consider basically supersymmetric moduli stabilization for both the complex structure and Kähler sectors, all eigenvalues of the mass-squared matrix are expected to satisfy the Breitenlohner-Freedman bound [10, 11]. However, for the purpose of uplifting to de-Sitter vacua (where the uplifting will be exponentially small) later, it is motivating to investigate the probability that all of the eigenvalues are positive at the supersymmetric vacua. Let us consider the Hessian

$$H_{IJ} = \partial_{\psi_I} \partial_{\psi_J} V|_{\text{extremal}}, \tag{17}$$

where ψ_I runs through x, s, u_i. Note that the extremum solution is inserted after taking the derivatives. Inserting the numerical solutions into the Hessian (17), we can

$h^{2,1}$	1	5	10	15	20	25
Probability	0.897	0.981	0.984	0.989	0.990	0.994

TABLE I: The probability of having a positive Hessian ($\partial_i \partial_j V$) at $h^{2,1} = 1, 5, 10, 15, 20, 25$. The probability is approaching unity as $h^{2,1}$ increases.

check the probability of positively defined mass squared matrix since the positivity of Hessian is the necessary and sufficient condition for the positivity of mass squared matrix at extremal points, due to the linear transformation.

Now we calculate the probability of having a positive Hessian in our model assuming the hierarchical setup. The result is shown in TABLE. I. This probability actually increases as $h^{2,1}$ increases, though the value at $h^{2,1} = 10$ is already very close to unity. Note that this probability is insensitive to the cutoff for b_i, d_i.

The reason is actually quite simple. Before introducing the non-perturbative term, we can estimate how the Hessian behaves by first restricting to the complex structure sector. At this stage, since we have the exact no-scale structure, the potential is simplified to be $V = e^K(|D_S W|^2 + |D_{U_i} W|^2)$ which is positively defined and bounded below. Then the supersymmetric solutions sit at Minkowski vacua, and the eigenvalues of Hessian are by definition semi-positive since the potential is convex downward at supersymmetric points. Next let us introduce the non-perturbative term. The correction to the supersymmetric solutions for the complex structure sector is of order of

$$0 \geq \frac{Ae^{-x}}{W_0} = \frac{1}{-(2x/3+1)} \sim \frac{3}{2\ \ln(-W_0/A)} > -1, \tag{18}$$

where we used (3), (5) for small $-W_0/A$, and $x > 0$. Note that the Kähler modulus stabilization (3) requires $0 \leq -W_0/A < 1$. In fact, the distribution of W_0 is more peaked toward $W_0 = 0$ as $h^{2,1}$ increases, therefore smaller values of W_0 become more likely [9]. Thus we see that the correction due to the non-perturbative term for Kähler modulus stabilization gets smaller and becomes negligible when we increase $h^{2,1}$. This is the reason why the positivity of the Hessian for large $h^{2,1}$ is satisfied at most of extremal points. This also means that the hierarchical structure between the complex structure sector and the Kähler sector we have employed for effective analysis is actually reliable, as anticipated in [1].

This behavior of the probability is compatible with that studied in [12], since their Gaussianly suppressed probability result will be applicable to our model only if we increase the number of Kähler moduli (lighter fields) to a large number. That a positive mass matrix is almost automatic for large $h^{2,1}$ is very encouraging in the search of meta-stable de-Sitter vacua in the KKLT models.

Combining with the result in [9], a clear preference for a vanishingly small Λ, either positive or negative, emerges. This is a consequence of the non-trivial functional dependences of Λ on the flux parameters, when probability theory is applied to these string theory scenarios. A better understanding of the distributions of the flux parameters will lead to a better determination of the likely values of Λ. It will be very interesting to find out what the likely value of Λ will be when we include additional interactions into the model (1) or consider other stringy models. The present work opens the door for further fruitful studies.

ACKNOWLEDGEMENTS

We benefitted from stimulating discussions with X. Chen, L. McAllister, F. Quevedo, Y. Wang, and T. Wrase.

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